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Understanding and overcoming the "positive profits with negative surplus-value" paradox

Entendendo e superando o paradoxo dos "lucros positivos com mais-valia negativa"

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RESUMO: Este artigo provê uma explicação do significado econômico do famoso paradoxo dos "lucros positivos com mais-valia negativa", apresentado por Steedman (1975), e demonstra que apesar dos valores-trabalho individuais poderem ser negativos em alguns sistemas de produção conjunta, a afirmação de que o trabalho incorporado no excedente da economia (i.e.,, mais-valia) é negativa baseia-se em pressupostos sem sentido (como níveis de atividade negativos). Oferecemos também uma forma de calcular a mais-valia de sistemas de produção conjunta que supera o problema e restabelece a proposição que afirma que trabalho excedente positivo é condição necessária para a existência de lucros positivos.
PALAVRAS-CHAVE: mais-valia; valor-trabalho; taxa de lucro; produção conjunta.

ABSTRACT: This paper explains the "positive profits with negative surplus-value" example of Steedman (1975) and shows that while in joint production systems individual labour values can be negative, the claim that the total labour embodied in the surplus product of the economy (surplus-value) can also be negative is based on assumptions that have no economic meaning (such as negative activity levels).The paper also provides a way to measure the surplus-value of joint production systems which overcomes the problems of the traditional concept and restates the proposition that a positive amount of surplus labour is a necessary condition for positive profits.
KEYWORDS: surplus value; labour value; rate of profit; joint production.
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INTRODUCTION

Steedman (1975) gave an example of a pure joint production system (i.e.,, without fixed capital or land) in which the rate of profits and their associated prices of production were positive and yet aggregate "surplus value" was negative. This example sparked a controversy related to the questions of how to define labour values in the context of joint production and particularly whether or not it implied a refutation of the so-called "fundamental Marxian theorem". This "theorem" states that a positive "rate of surplus value" – the ratio between the quantity of embodied labour on the physical surplus ("surplus value") and the quantity of embodied labour in the necessary consumption of workers ("variable capital") – is a necessary and sufficient condition for a positive general rate of profits.

Although the literature on the possibility of *individual* negative labour-values quite exhaustively scrutinized the necessary and sufficient conditions for it to happen – see Schefold (1989), Filippini & Filippini (1982) and Hosoda (1993) – the same cannot be said on the literature related to the possibility of negative surplus-value – i.e.,, the labour-value of the aggregate surplus product of the economy. Many authors tried to get around Steedman's critique through redefinitions of the labour values of single commodities, arguments about heterogeneous labour and considerations about the problem of choice of technique. Few contributors, however, discussed the actual economic meaning and relevance of the special (implicit) assumptions in Steedman's example.

In this paper we do that by means of a critical survey and a theoretical evaluation of this literature[1]. We make use of Steedman's original numerical example as basis to compare and contrast the different contributions using the activity analysis diagram[2] which provides a very clear picture of some properties of a square two-commodity system[3].

Our main conclusions are that although labour values for some single commodities can indeed be negative in the most general case (it *may* happen if the system is not "all-productive", as defined by Schefold (1978)), the basic and sensible classical proposition that in the aggregate less labour would be necessary to produce only the necessary wage basket than to meet also the total final demand coming from the expenditures of the capitalists always holds, even if in some cases (as in the particular economic system depicted by Steedman) producing the neces-

[1] We decided to discuss the so-called "New Solution" approach as potential solution to the paradox in an appendix because it was not part of the original controversy.

[2] Originally presented by Dorfman, Samuelson and Solow (1958).

[3] It is important to note that the all the contributions presented here hold for any number of commodities. The use of the original example is just to facilitate the exposition. For instance, the graphical device could not be used at all in a system with more than three commodities and even in the case of three commodities would only makes things more complicated without adding any new conclusion.

sary wage basket could entail also jointly producing some extra outputs unneeded by the workers.

Besides scrutinizing the original example and the controversy, we also propose a reformulation of a method originally presented by Akyüz (1983) and with that it will be shown that it is always possible to find an economically meaningful amount – i.e.,, in the precise sense that contains only feasible (non-negative) levels of activity – of aggregate surplus-value for the economy without changing the processes in use (i.e.,, using the dominant techniques in use in the square system). As it will be seen, this method seems to be the only available one which is coherent with the classical-marxian idea of taking as given the processes of production in use to measure the aggregate amount of extracted surplus labour.

The paper is organized as follows. Second section briefly discusses issues related to labour values in the context of joint production since the original contribution of Sraffa [1960]. Third section presents and discusses Steedman's assumptions and results. Fourth section reviews the debate on positive profits and negative surplus value. Fifth section offers concluding remarks.

LABOUR VALUES AND JOINT PRODUCTION

Labour values, in the case of homogeneous labor, are the physical quantities of labour directly and indirectly necessary to produce a unit of a particular commodity using the dominant (or socially necessary) methods of production actually in use. Sraffa (1960) has shown, for single production systems, that the set of prices of production measured in labour commanded (i.e.,, divided by the money wage) coincide with the set of labour values when the rate of profits is zero.

In the general case of joint production the author points out that there is naturally an obvious difficulty in thinking of what is the quantity of labour directly and indirectly necessary to produce a particular single commodity, since more than one commodity can be produced by the same process and at the same time that same commodity may be produced by other processes of production.

For the cases where it is possible to produce single commodities separately, i.e.,, to increase the net output of a particular commodity without necessarily increasing the net output of any other[4], it is also possible to calculate precisely what is the quantity of labour directly necessary to produce only one unit of that particular commodity and, thus, there is no difficulty in calculating its positive and additive labour value. That seems to be why simple systems with non-shiftable fixed capital or the analysis of land and differential rents tend to behave like single product systems. In general

[4] These systems are variedly known as "all-productive" (Schefold, 1978), or having "weak joint production" (Abraham-Fois and Berrebi, 1997), or having the "adjustment" property, and by adding the assumption of constant returns to scale also, as having the "non-substitution" property (Bidard and Erreygers, 1998).

the necessary (but not sufficient) condition for non separability is that there are processes in use that produce net outputs of more than one commodity.

However, things get more complicated within pure joint product systems that do not possess the property of separability. The difficulties with the concept of labour value of individual commodities arise exactly because of this reason in these systems.

For these type of systems Sraffa (1960, p.59) mentions the possibility that the labour values of individual commodities could turn out to be negative. The author explains the meaning of such negative labour values by referring to the fact that labour values are one and the same thing as the employment multipliers of a single commodity associated to that system of production. A negative labour value thus means that if we think of increasing the net product of only that particular commodity by one unity, and production is not separable and thus we will also increase the net output of at least one other commodity, we will necessarily have to increase the amount of labour employed by one process but also will have to decrease the level of employment in at least some other process to prevent the overproduction of the second commodity.

Now depending on the direct amount of labour employed in the process that is expanded compared with the process that is contracted, it may happen that in the end the total amount of labour employed in all processes will be lower than it was before the increased of the net output of the first commodity. In this case the commodity in question will have a negative labour value because an increase in its production has led to a decrease in total (direct and indirect) employment.

Sraffa (1960) also warned in a footnote[5] of the possibility of the awkward occurrence of what he called "negative industries" by which he meant that the fact that since under joint production the activity levels of some processes must be decreased when that of the others increase to match the level of composition of the "requirements of use" was logically possible, if the change in the level and structure of demand was sufficiently drastic, that the contraction of some particular joint processes of production could be so extreme as to require it to be "activated" at a negative level. Since negative activity levels do not exist, this only means that in fact that even contracting these processes to zero the net output of the overabundant products would not be reduced sufficiently for the system to adjust the vector of production exactly to the vector of effectual demands – in this case, the processes in use would have to change in order to match exactly the vector of effectual demands.

STEEDMAN'S "POSITIVE PROFITS WITH NEGATIVE SURPLUS-VALUE" PARADOX

Steedman (1975) seems to have taken this route and, by inverting the assumptions that Sraffa used to rule out processes with negative activity levels, produced

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[5] Sraffa (1960, footnote 1, p. 57).

his own particular example. The inconclusive results of the controversy sparked by his "positive profits with negative surplus value" example seem to have been due to the combination of two main factors. The first is that neither much explanation of the economic meaning of the original results nor of the relevance of its assumptions has ever been made[6]. The second reason is that, with very few exceptions, his critics seemed so concerned to "defend" the concept of surplus value and/or the labour theory of value in some form that they tended mostly to add further ad hoc assumptions to those of Steedman's example in order to change its conclusions, instead of scrutinizing the nature of the example and its implications.

Steedman (1975, p. 115) assumes an economy capable of producing a surplus of two commodities using two joint production processes that use only circulating capital:

INPUTS							**OUTPUTS**		
commodity 1		commodity 2		labour		commodity 1		commodity 2	
5	$\oplus$	0	$\oplus$	1	$\rightarrow$	6	$\oplus$	1	
0	$\oplus$	10	$\oplus$	1	$\rightarrow$	3	$\oplus$	12	

He also assumes that the wage is paid at the end of the production period and that the real wage basket is exogenously given: $\mathbf{b} =$ (0.5, 0.83). This assumption is not necessary for the present purposes – it does not matter for the results if wages are paid ex-ante or ex-post.

The really crucial assumptions implicit in his analysis amount to three:

(i) A two commodities square system in which both processes generate the same joint products.

(ii) There is a process which is strictly superior to the other one, the second one, having higher net products for both goods[7]. In Steedman's original example we have the following net products of each process (operated at unity levels): $\mathbf{m}_1$=(1, 1) and $\mathbf{m}_2$=(3, 2), the columns of the matrix (**B-A**).

These two assumptions imply that the *individual* labour value for the good one is negative. The third assumption is:

(iii) The proportions in which commodities one and two are demanded by the capitalists are very different from the proportions in which they appear in the wage basket.

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6 According to us, there has been too much focus on the fact that one process strictly dominates the other. As we will see, this simple fact is far from being the only relevant assumption.

7 Wolfstetter (1976) has pointed out that in a two-commodity model, negative individual labour-values can happen if, and only if, there is a strictly inferior process. However, Hosoda (1993) shows that in models with more than two-commodities, individual negative labour values can occur even without the presence of strictly inferior processes, as defined by Wolfstetter (1976).

This will mean that no combination of the two processes available is capable of producing without overproduction either only the wage basket or only the profit earner´s basket. As we shall see it is this latter assumption of the unfeasibility of producing the baskets of the two classes separately plus the great divergence among the bundles, not the mere existence of a negative labour value for one of the commodities, which is the key for the occurrence of negative *aggregates* of labour.

Steedman (1975, p. 115) first shows that in this particular joint production system the rate of profits and relative prices of production are positive, since the profit rate depends only on the existence of a positive net product (surplus product) after the deduction of the necessary consumption for being positive – independently of the labour value measurement.

He then shows that one of the two commodities has a negative labour value:

$$\begin{aligned} 5\lambda_1 + 1 &= 6\lambda_1 + \lambda_2 \\ 10\lambda_2 + 1 &= 3\lambda_1 + 12\lambda_2 \end{aligned} \quad (1)$$

Where λ_1 and λ_2 are the labour values of each commodity. The solution of this system gives us the following vector of labour values:

$$\Lambda = \mathbf{a}_L(\mathbf{B} - \mathbf{A})^{-1} = (\lambda_1, \lambda_2) = (-1,2) \quad (2)$$

Where $\mathbf{a}_L$ is the vector of direct labour requirements (composed only by units, given the normalization presented in the table above) and Λ is the vector of labour values (which also corresponds to the relative prices associated to a profit rate equal to zero, using the wage rate to normalize them).

He then moves on to show also that the aggregate amount of surplus value in this system is negative, both measured as the aggregate labour value of the sum of commodities demanded by the capitalists and as the difference of total labour employed versus the aggregate labour employed (embodied) in the wage basket.

The total net product is y is the sum of each class bundle $\mathbf{y}_k = (5, 2)$ which is the final demand which is appropriated by the capitalists, and, $\mathbf{y}_w = (3, 5)$, the workers' consumption.

To produce this final demand

$$\mathbf{y} = (\mathbf{B} - \mathbf{A})\mathbf{x} = (8,7) \quad (3)$$

a feasible combination of both processes is required:

$$\begin{aligned} x_1 + 3x_2 &= 8 \\ x_1 + 2x_2 &= 7 \end{aligned} \quad (4)$$

Where x_1 and x_2 are the levels of activity of each process required to produce the net product (the components of the vector of activity levels). The solution of this system gives the following vector:

$$\mathbf{x} = (\mathbf{B} - \mathbf{A})^{-1}\mathbf{y} = (5,1) \tag{5}$$

The aggregate amount of labour is given by the sum of each element of the activity level vector using this normalization:

$$L = \mathbf{a}_L\mathbf{x} = x_1 + x_2 = 6 \tag{6}$$

If we multiply the final demand baskets of each class by the labour values of each commodity, this particular technology and pattern of final demand will give us the following labour value aggregates:

$$\begin{aligned} V &= \Lambda \mathbf{y}_W = \mathbf{a}_L (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y}_w = (-1).3 + (2).5 = 7 \\ S &= \Lambda \mathbf{y}_k = \mathbf{a}_L (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y}_k = (-1).5 + (2).2 = -1 \end{aligned} \tag{7}$$

Where S is the surplus value and V is the variable capital. The other way for calculating the surplus value is looking at the difference between the living labour (6 units) and the variable capital (7 units):

$$S = L - V = 6 - 7 = -1 \tag{8}$$

Before moving on, we think that we can understand this result better if we make the calculation in terms of activity levels associated to the production of each bundle, instead of just finding the labour value of each bundle, as we have done for the whole net product above. For the workers' bundle

$$(1) \quad \mathbf{y}_W = (\mathbf{B} - \mathbf{A})\mathbf{x}_W = (3,5) \tag{9}$$

we need to solve:

$$\begin{aligned} x_{w1} + 3x_{w2} &= 3 \\ x_{w1} + 2x_{w2} &= 5 \end{aligned} \tag{10}$$

Which gives us the following "activity levels" required to produce workers' consumption:

$$\mathbf{x}_w = (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y}_w = (9,-2) \tag{11}$$

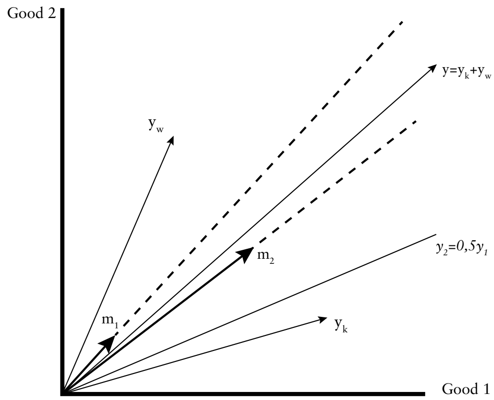
The sum of each component of $\mathbf{x}_w$ gives us the variable capital. Applying the same procedure for the capitalists' bundle we get:

$$\mathbf{x}_k = (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y}_k = (-4,3) \tag{12}$$

Thus, the surplus value S can be calculated from the sum of the components $\mathbf{x}_{k.}$

This way of calculating shows that both bundles require negative activity levels of one of the processes because both bundles cannot be produced *separately* by the processes in use.

Grafically, we have that:



The dotted lines provide the set ("cone") of feasible net products using the two processes given by the vectors $\mathbf{m}_1$ e $\mathbf{m}_2$. The horizontal axis represents the net product of first commodity and the vertical axis the the net product of the second commodity.

As we can see, the total final demand **y** falls inside the cone, which means that it can be reached with non-negative levels of activity. The total employment required for that is 6 units. However, the bundles that each class receives fall outside the cone. This means that they cannot be produced *separately* – though it is feasible to produce both jointly. The consequence of this is that negative levels of activity would be required to produce only workers' consumption or only the final demand of capitalists. If both bundles were inside the cone no negative surplus value could happen, although the labour value of one of the two commodities would still be negative. Indeed, with the same aggregate final demand **y**=(8, 7) but with the consumption of the workers being y_w=(3, 2.5) and the expenditures of the capitalists being $\mathbf{y}_k$=(5, 4.5), for example, we would still get, overall activity levels **x**=(5, 1) and total employment L=6. But now $\mathbf{x}_k$=(3.5, 0.5) and thus S= 3.5+0.5=4 and surplus value would be positive. For the workers we would have $\mathbf{x}_w$ = (1.5, 0.5) and V=2 and thus S= 6- 2=4.

In the example, for the capitalists' bundle we have that the total employment required to produce it is negative, i.e.,, negative surplus value. So, we may say that within the group of final demands that are non-feasible, there is a group which "requires" negative *amount* of employment to be produced, which in this numerical example[8] are those ones that lie below the line y_2=0,5y_1. But the central point we want to emphasize is that any final demand outside the cone is economically meaningless[9].

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[8] The sufficient conditions for a bundle to "need" negative aggregate amount of employment in a general two commodity system (that is, being below the line in the graph) and its economic meaning are derived in the Appendix I.

[9] For example, if the final demands were y_k=(5, 3) and y_w=(3, 4), we would have positive values for S and V, the same total employment of 6 units and it would seem that no paradox happens. However,

We can thus see that the existence of negative labour values for single commodities does not by itself imply the paradox. The paradox of positive profits and negative surplus value provided by Steedman (1975) rests on the fact that negative levels of activity would be required if the bundles were to be produced separately – or, that these bundles can only be produced jointly within this square system.

Of course negative activity levels do not exist. They are just the mathematical symptom of the fact that there would be overproduction of one of the two commodities if we were to produce only the wage bundle of the workers using these two processes. So the right conclusion from Steedman's example is that in general joint production systems the calculation of aggregate surplus labour should be adapted to deal with the possibility of notional overproduction, instead of using the economically meaningless notion of negative activity levels.

SURPLUS VALUE AFTER STEEDMAN

An initial reaction to the Steedman's example came from Morishima (1976). Based on his earlier works such as Morishima (1973) and Morishima (1974), the author proposed to redefine the labour values in joint production using what he calls the "true values". The "true value" of a commodity (or a bundle) is given by the minimization of labour-time required to produce one net unity of it. Be $\mathbf{e}_i$ the column vector in which the i-th coordinate is unity and the other ones are null, the true-value of the commodity i will be given by the following minimization problem:

$$\begin{aligned} &\text{min. } \mathbf{a}_L\mathbf{x} \\ &\text{s.t. } \mathbf{Bx} \geq \mathbf{Ax} + \mathbf{e}_i,\ \mathbf{x} \geq \mathbf{0} \end{aligned} \qquad (13)$$

Where activity levels and the processes in use are the endogenous variables of the problem[10] and the symbol ≥ means that the vector is equal (in every coordinate) or higher (in at least one coordinate) than the other one. To find the "true variable capital" $\mathbf{e}_i$ must be substituted by $\mathbf{y}_w$, which is going to be the total employment required to produce workers' consumption which minimizes the amount of labour expended.

Be $\mathbf{x}_w$* the activity levels that solves the problem, the "true variable capital" would be:

$$V^* = \mathbf{a}_L\mathbf{x}_w^* \qquad (14)$$

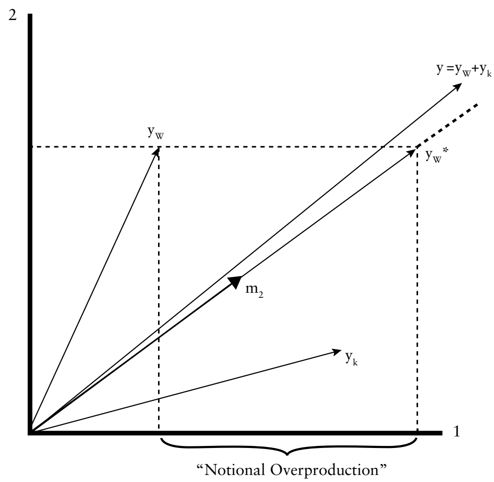
both bundles fall outside the cone – activity levels would be x_k=(-1, 2) and x_w=(6, -1) – which means that, despite being positive, either variable capital or surplus value have no economic meaning.

[10] Matrixes A and B may not be square in the general case discussed by Morishima (1974).

And the "true surplus value":

$$S^* = L - V^* \tag{15}$$

Using Steedman's example, graphically it would be:



Where $\mathbf{y}_w^*$ is the net product associated with the level of activity $\mathbf{x}_w^*$.

In this case, only the process with higher productivity would be operated (this is why $\mathbf{y}_w^*$ lies above the same line as $\mathbf{m}_2$). Because operating this process alone cannot produce exactly the workers' consumption, there would be excess production of the first commodity of 4,5 units. But the point is that less labour than the total employment would be required to produce workers' consumption. To produce $\mathbf{y}_w$=(3, 5) it would be necessary to operate only the second processes with 2,5 units of labour. This gives a positive "true surplus value" of 6-2,5=3,5.

The criticisms of this procedure are not new: "true-values" are not additive like in Marx, i.e.,, the "true-value" of a bundle of commodities is not equal to the sum of "true-values" of the same commodities separately produced. The authors argue that this different definition of labour value would have a textual basis in Marx but the argument is not very convicing (Steedman, 1976a). A second important criticism is that in this case the (redefined) surplus value would be related to a non-capitalist (i.e.,, nonprofit-maximizing) criterion for the choice of technique – while in Marx and in the literature related to the "fundamental marxian theorem" this was based on the processes in use in capitalism ("the socially necessary techniques") (Akyüz, 1983). In fact the "true value" calculations measure the hypothetical minimum amount of labour that would be necessary to produce the wage basket for society as such and not the amount that is "socially necessary" given the techniques and processes actually already chosen by a capitalist criterion of choice of technique.

Wolfstetter (1976) questioned Steedman's use of Sraffa's approach of assuming square joint production systems with equalities, instead of starting from

Von Neumann's method of rectangular systems and using inequalities. Steedman (1976b) promptly replied that in the particular case of his example the difference between the two methods would hardly matter for his results, the solution being exactly the same for both approaches. Moreover, Wolfstetter's third theorem (Wolfstetter, 1976, p. 867), which states that the existence of an inferior process is a necessary and sufficient condition for negative individual labour values, is correct only in a two commodity system as pointed out in the footnote 3 above.

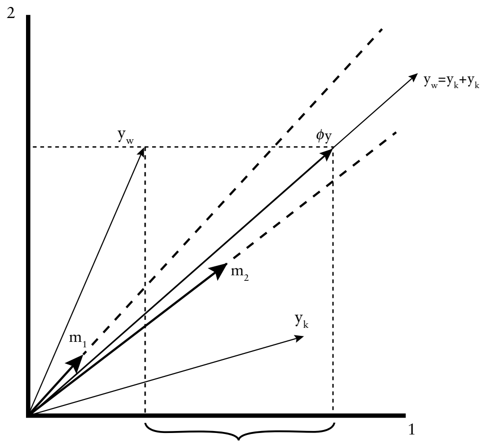
Kurz (1979) and Krause (1980) tried to refute the paradox redefining the vector of individual labour values to make it semi-positive. Krause (1980) argued that the different labour productivities are a case of heterogeneous labour and with this argument the author changes the weights of the direct labour vector in such way to guarantee strictly positive individual labour values. The obvious problem with this complicated solution is that the assumption of heterogeneous labour is simply not present in Steedman (1975). Flaschel (1983, p. 437) adds relative prices to the definition of individual labour value in order to get positive magnitudes for the latter. This procedure goes in a completely different route with respect to the original function of the labour values of being a measure of physical costs, independent from (and a major determinant of) income distribution and relative prices.

Akyüz (1983) argues that we can always find $\mathbf{y}_v$ that is a fraction of the actual net product of the economy in such a way that it is enough to achieve the actual workers' consumption. That is, some measure of the 'variable capital' using the processes actually in use and keeping the same proportion in which they are actually operated can be found and used as the basis to calculate the surplus-value of more general joint production systems. He adds that using this method there's no need for solving a minimization problem and that this would be a great virtue of it (Akyüz, 1982, p. 171).

The problem is that in this case there are many scalars that satisfy the inequality and we would have an indeterminacy problem. It seems to us that the author's insight can be improved if presented as a minimization problem of finding the minimum scalar that satisfies the above conditions in the following way. This notional scale of production should be given by a scalar ϕ given by the following minimization problem:

$$\begin{aligned} &\text{Min } \phi \\ &\text{s.t. } \phi \boldsymbol{y} \geq y_w ; \phi \in (0,1) \end{aligned} \qquad (16)$$

Where $\mathbf{y}$ is the actual net product of the system and $\mathbf{y}_w$ is the actual workers' consumption basket. Graphically, we would have:



"Notional Overproduction"

As we can see that the vector falls inside the cone and, consequently, a positive amount of employment is required to produce it. In this case, we would have a redefined vector of workers' consumption which has the advantage of being based in the processes actually in use and keeping the same proportion in which they are actually operated. This redefined surplus value *S'* will be given by:

$$S' = (1 - \phi)\, L \tag{17}$$

If we make a comparison with the "true-values" we can see that this method requires much less new information and is much closer to the actual system data than the former. In the procedure proposed by Morishima (1976) the processes and their relative levels of activity are altered (besides the presence of non-profit maximizing criterion for the choice of technique). In the method proposed by Akyüz (1983) there's only the scalar as a novelty and no change in the choice of techniques is required. Inevitably (as in the "true-value" approach) there's some excess of production – of the first commodity (2,174 units in the example).

Our improvement to Akyüz's method allows us to see that the correctness of the sensible classical proposition that (without changing the actual processes that are in use) in a capitalist economy workers obviously work more than they would need to produce only (at least) what is contained in their wage basket.

This proposition relating positive profits and positive surplus labour is of general validity with or without joint production, and whatever may be happening to the labour value of individual commodities. It is important to point out that this "notional overproduction" is not something that will or even may actually happen in this economy. The term means only that this excess *would* happen *if* the system were to produce these different quantities using the same techniques. But this is *not* the case and these calculations are just like Sraffa's standard system: *abstractions* that reveal some important properties of *the actual system* – in our case, the possibility of finding an economically meaningful measure of aggregate surplus value. The calculations only provide a useful piece of information: how much labour would be needed to produce with minimum waste the workers' consumption basket.

More recently Trigg and Philp (2008) have claimed that there will always be positive surplus value in the aggregate – even in the presence of individual negative labour values for single commodities – using the argument that Kahn's multiplier is positive and it depends only on the fact that the value of the labor power is below one and this will always be true for systems that produce positive surplus product. In order to understand their point, we have to start with the gross product decomposition

$$\mathbf{Bx} = \mathbf{Ax} + \mathbf{ba}_L \mathbf{x} + \mathbf{y}_k \tag{18}$$

This can be rewritten in the present case as

$$\mathbf{y} = \mathbf{ba}_L (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y} + \mathbf{y}_k \tag{19}$$

Which can be rewritten as

$$\mathbf{y} = \mathbf{b} \Lambda \mathbf{y} + \mathbf{y}_k \tag{20}$$

Pre-multiplying both sides by the labor-value vector we get

$$\Lambda \mathbf{y} = \Lambda \mathbf{b} \Lambda \mathbf{y} + \Lambda \mathbf{y}_k \tag{21}$$

Thus,

$$L = \left(\frac{1}{1 - \Lambda \mathbf{b}} \right) \Lambda \mathbf{y}_k \tag{22}$$

Thus, from a Kahn-type multiplier perspective, we can see the equation in the following way: total employment is a multiple of the employment required to produce capitalists' final demand. This multiple – the Kahn "multiplier" – depends on the value of the labour power ($\Lambda \mathbf{b}$ in the denominator). Thus, the authors argue, it is enough the value of labor power is smaller than one for the multiplier to be greater than one and positive even if some individual labour-values are negative. Hence, according to them, positive surplus value is always guaranteed.

The problem with this argument is that its crucial assumption does not always hold. In fact, in Steedman's example the value of the labour power is greater than one (to see that, just divide the variable capital by the total living labour) because of the non-separability problem discussed before. Thus, the argument put forward by Trigg and Philp (2008) that Kahn's multiplier is positive – and that this implies positive surplus-value – for any system able to produce surplus even in the case of negative individual labour-values is correct only under the case where both final demand vectors fall inside the cone of feasible sets.

CONCLUDING REMARKS

The present work has proposed to give a clarification of the economic properties behind the "positive profits with negative surplus value" example provided by Steedman (1975). Criticisms of this example such as the lack of economic meaning of the presence of an inferior process seem quite misleading because the negative marxian aggregates are not a direct consequence of it[11].

Individual negative labour values are certainly a necessary condition for the paradox to happen but not sufficient. The central point seems to be the fact that the bundles that go to each class must: (i) fall outside the cone given by the square system and (ii) have a very different composition. So, the negative surplus value of the example is related to the fact that negative levels of activity would be "required" to produce only (with no overproduction) the capitalists' bundle – something that is devoid of economic meaning. Hence the "phenomenon" of negative surplus value *does not* mean that in order to produce only workers' consumption more labour would be required than in the case where there is also the capitalists' consumption. It only means that the workers' consumption bundle *cannot* be produced separately by this joint production system and, *thus, there would be some overproduction if there were no demand coming from the capitalist* (i.e.,, if the bundle were produced separately).

As the alternative provided by Morishima's true values does not seem useful as being too "normative" we think that our improvement of the method originally proposed by Akyüz (1983) seems to be the simplest and at the same time least arbitrary way to get an economically meaningful and thus positive measure of surplus labour in viable pure joint production systems. The procedure has the advantage of using only data from the processes that are actually in use as opposed to the literature on the so called "fundamental Marxian theorem".

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[11] As pointed out in the footnote 4, although the existence of inferior process is a sufficient condition for individual negative labour values to occur (in 2x2 is a necessary and sufficient condition), it may happen also in other cases (Hosoda, 1993).

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APPENDIX I: THE ECONOMICS OF THE UNFEASIBLE BUNDLES IN A TWO-COMMODITY SYSTEM

As mentioned before, to generate negative surplus value (or, more generally, any negative aggregate of labour value) more than unfeasible bundles are required. Although negative levels of activity are a necessary condition it is also necessary to assume a very specific composition of a bundle **y** to obtain the paradox. In the very simple algebraic demonstration that follows we will show this further condition in the two commodity two processes case. The proof for the general case of any dimensions will be left for another occasion.

As we have normalized the system for units of direct labour, we have that the labour value of a bundle **y** is given by:

$$\Lambda \mathbf{y} = \mathbf{a}_L (\mathbf{B} - \mathbf{A})^{-1} \mathbf{y} = \mathbf{a}_L \mathbf{x} = x_1 + x_2 \qquad (23)$$

Let us define **M** such as:

$$\mathbf{M} = (\mathbf{B} - \mathbf{A}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{24}$$

the inverse of **M** in this case is given by

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \tag{25}$$

Provided that det**M** is non-null, we know that:

$$\mathbf{x} = \mathbf{M}^{-1}\mathbf{y} \tag{26}$$

If we want a negative labour value of a bundle **y** to occur we need to have the following condition

$$x_1 + x_2 = (y_1 m_{22} - y_2 m_{12}) + (-y_1 m_{21} + y_2 m_{22}) < 0 \tag{27}$$

This gives the following inequality:

$$\frac{y_1}{y_2} > \frac{m_{12} - m_{11}}{m_{22} - m_{21}} \tag{28}$$

Which shows that for having a negative amount of employment necessary to "produce" a bundle we need to assume that the ratio of the two commodities in a bundle is greater than the ratio of the differences of efficiency in the production of commodities of process one relative to process two. This means that only the existence of a strictly superior process is not sufficient to obtain a negative aggregate labour value of a bundle of commodities. We need either a very wide disparity in the demand for good one and two and/or a small difference of efficiency between the two processes.

APPENDIX II: IS THE "NEW SOLUTION" A SULUTION FOR THE PARADOX?

Foley (1982) and Duménil (1983) have proposed a "New Interpretation" for the Marxian theory of value – which later has been called the "New Solution" to the "transformation problem". The original contribution of these authors was related only to single production and, as far as we know, has never been applied to the paradox we discussed before. However, given its popularity nowadays, it is worth discussing it within the present case. Since it has never been part of the original controversy, however, we have chosen to put it in an appendix.

The "New Solution" proposes a new interpretation for the original Marx's variables. Rather than deriving the Marxian aggregate labor values from the technology and the historical living standard workers' consumption, these authors propose to redefine the original concepts in the following way: the money wage (or the wage bill) and the "value of labor power" (or, alternatively, the "rate of exploitation") are exogenously given[12]. If, besides it, we normalize the system as

$$\mathbf{py} = L \tag{29}$$

Then, by assumption, the value of the net product equals the labor value of the net product and, these authors claim, one of the Marxian invariance postulates is restored – although the original Marx's equality was with respect to the gross product. Moreover, we can write the net product as

$$\mathbf{py} = wL + r\mathbf{pAx} = W + \Pi \tag{30}$$

The exogenously given money wage normalized equals the labor share

$$v_{NS} \equiv \frac{w}{\frac{\mathbf{py}}{L}} \tag{31}$$

If we call it "value of labor power" v_{NS}, we obtain the equality between profits and surplus value:

$$(1 - v_{NS})L \equiv S_{NS} = \Pi \tag{32}$$

We will not discuss here if this is a "solution" or even if there was ever a "transformation problem", since our point is related to Steedman's paradox and the possibility of calculating the aggregate rate of exploitation. Since the 'value of the labor power' – i.e.,, the labor share in the national income – is exogenous, once it is postulated to be positive (between 0 and 1, as can be seen from national accounts), then the "rate of exploitation" – i.e.,, the ratio between mass of profits and the wage bill – will necessarily be positive since

$$e_{NS} \equiv \frac{1}{v_{NS}} - 1 \tag{33}$$

Thus, positive "surplus value" will always occur if profits are positive, which is guaranteed if the technology is able to produce surplus.

Does it solve the original problem? Since the original problem was not even

[12] Commas will be used to highlight that the traditional variables have different meaning and the index "NS" will be used to represent these redefined variables, in order to avoid confusion with the original concepts used in the article.

mentioned here – the problem of finding invariant aggregates with respect to the distribution, from the technology and the real wage data, which are related to the price aggregates – the answer is negative. What is called "surplus-value" or 'rate of exploitation" here is simply the money values normalized in a specific way. Actually, there is not even the possibility to occur any paradox between the "physical" (i.e.,, technology and real wage bundle) and the price magnitudes, because the former play no role in the new definitions.

Thus, in our opinion, by definition, it is spurious to claim that this approach could be argued to solve Steedman's paradox. Besides, we think that the term "New Interpretation"[13], as it was originally used by its proponents, seems to be much more appropriate – since it is conceptually different than the traditional view on this issue – than the unfortunate "New Solution", as popularized later[14].

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[13] On the dubious conceptual legitimacy of the New Solution, see Petri (2015).

[14] It is important to observe that for some joint production systems the same exogenous money wage and "rate of exploitation" – i.e.,, the New Solution exogenous variables – may give rise to more than one set of positive profit rates and positive prices – that is, the relative price system becomes indeterminate and the approach does not provide any economic criterion to choose among them (Steedman, 1992). Thus, even taking for granted that it could be a solution from a conceptual perspective, this approach would not be devoid of problems.